$$Q_2 = (\theta)_{x=0}(\partial H/\partial q_2)_{x=0} \tag{12}$$

Using Eqs. (6) and (10), we get

$$V = (\tfrac{1}{6})cq_1{}^2q_2 \tag{13}$$

$$\partial V/\partial q_2 = (\tfrac{1}{6})cq_1{}^2 \tag{14}$$

The time rate of change of heat flow obtained from Eq. (8), is

$$\dot{H} = (\tfrac{1}{2})c(q_1\dot{q}_2+\dot{q}_1q_2)(1-x/q_2)^2+cq_1\dot{q}_2(x/q_2)(1-x/q_2)+\rho L\dot{q}_2 \tag{15}$$

and

$$\partial H/\partial q_2 = (\tfrac{1}{2})cq_1(1-x/q_2)^2+cq_1(x/q_2)(1-x/q_2)+\rho L \tag{16}$$

Substituting Eq. (15) in Eq. (11) and Eq. (16) in Eq. (12), it gives

$$D = (1/2k)[(c^2q_2/20)(q_1\dot{q}_2+\dot{q}_1q_2)^2+(c^2q_1{}^2q_2\dot{q}_2{}^2/30)+\rho^2L^2q_2\dot{q}_2{}^2+(q_1\dot{q}_2+\dot{q}_1q_2)(c^2q_1q_2\dot{q}_2/20+c\rho Lq_2\dot{q}_2/3)+c\rho Lq_1q_2\dot{q}_2{}^2/3] \tag{17}$$

$$\partial D/\partial\dot{q}_2 = (q_2\dot{q}_2/2k)(2\rho^2L^2+4c\rho Lq_1/3+4c^2q_1{}^2/15)+(q_2{}^2\dot{q}_1/2k)(3c^2q_1/20+c\rho L/3) \tag{18}$$

and

$$Q_2 = (cq_1{}^2/2)+\rho Lq_1 \tag{19}$$

The Lagrangian equation, Eq. (9), in the generalized coordinate q_2, becomes

$$(q_2\dot{q}_2/2k)(2\rho^2L^2+4c\rho Lq_1/3+4c^2q_1{}^2/15)+(q_2{}^2\dot{q}_1/2k)(3c^2q_1/20+c\rho L/3) = cq_1{}^2/3+\rho Lq_1 \tag{20}$$

Using Eq. (6), the boundary condition, Eq. (4), is given by

$$h(T_f-q_1)-k(q_1/q_2) = 0 \tag{21}$$

Use of this boundary condition, Eq. (21), at the surface, provides an additional relation between the generalized coordinates and thus reduces by one the number of differential equations of the Lagrangian type. We now nondimensionalize Eqs. (20) and (21) by defining the following quantities

$$\tau = h^2T_f t/\rho Lk$$
$$\eta = hq_2/k$$
$$\beta = cT_f/\rho L$$
$$\Phi = q_1/T_f$$

Equation (21), in the dimensionless form, gives the surface temperature

$$q_1/T_f = \Phi = \eta/(1+\eta) \tag{22}$$

Equation (20), in the dimensionless form, after simplification, becomes

$$\dot{\eta}[A_1\eta^3+A_2\eta^2+A_3\eta+A_4] = B_1\eta^2+B_2\eta+B_3 \tag{23}$$

where

$$A_1 = 16\beta^2+80\beta+120 \qquad B_1 = 40(\beta+3)$$
$$A_2 = 25\beta^2+180\beta+360 \qquad B_2 = 40(\beta+6)$$
$$A_3 = 100\beta+360 \qquad B_3 = 120$$
$$A_4 = 120$$

Solution of Eq. (23) gives the melting rate

$$\tau = (A_1/2B_1)\eta^2+(A_2-A_1B_2/B_1)(\eta/B_1)+[(1/2B_1)(A_3-A_1B_3/B_1)-(B_2/2B_1)(A_2-A_1B_2/B_1)]\times[\ln(B_1\eta^2+B_2\eta+B_3)/B_3]+[A_4+(A_2-A_1B_2/B_1)\times(B_2{}^2-2B_1B_3)/2B_1{}^2-(B_2/2B_1)(A_3-A_1B_3/B_1)]\times(2/(4B_1B_3-B_2{}^2)^{1/2})\times[\tan^{-1}(2B_1\eta+B_2)/(4B_1B_3-B_2{}^2)^{1/2}-\tan^{-1}(B_2)/(4B_1B_3-B_2{}^2)] \tag{24}$$

It can be easily seen that for very high latent heat ($L\to\infty$, $\beta\to 0$), Eq. (24) reduces to

$$\tau = (\eta^2/2)+\eta \tag{25}$$

This is the same equation as obtained by Goodman,[6] for the melting rate, using the heat balance integral method. Results for surface temperature–time history $\Phi(\tau)$ and the melting rate $\eta(\tau)$ for different values of β are plotted in Figs. 1 and 2, respectively, and compared with Goodman's solution.

It is concluded that Biot's variational method is applicable to the phase-change problem with aerodynamic heating. The simplicity of the method lies in using only a linear temperature profile, yielding satisfactory results.

References

[1] Lardner, T. J., "Approximate Solutions to Phase-Change Problems," *AIAA Journal*, Vol. 5, No. 11, Nov. 1967, pp. 2079–2080.

[2] Biot, M. A. and Daughaday, H., "Variational Analysis of Ablation," *Journal of the Aerospace Sciences*, Vol. 29, No. 2, Feb. 1962, pp. 227–229.

[3] Biot, M. A. and Agrawal, H. C., "Variational Analysis of Ablation for Variable Properties," *Transactions of the ASME, Ser. C: Journal of Heat Transfer*, Vol. 86, No. 3, Aug. 1964, pp. 437–442.

[4] Prasad, A., "Biot's Variational Principle in a Phase-Change Problem," Master's thesis, Oct. 1968, Dept. of Mechanical Engineering, Indian Institute of Technology, Kanpur, India.

[5] Prasad, A. and Agrawal, H. C., "Biot's Variational Principle for a Stefan Problem," *AIAA Journal*, Vol. 10, No. 3, March 1972, pp. 325–327.

[6] Goodman, T. R., "The Heat-Balance Integral and Its Application to Problems Involving a Change of Phase," *Transactions of the ASME*, Vol. 80, No. 3, Feb. 1958, pp. 335–342.

Compressibility Effects in Unsteady Thin-Airfoil Theory

Roy K. Amiet*

United Aircraft Research Laboratories, East Hartford, Conn.

Nomenclature

- b = semichord
- $C(k)$ = Theodorsen function
- $f(M)$ = function of Mach number, defined by Eq. (8b)
- g = kernel function defined by Eq. (7b)
- h = kernel function defined by Eq. (8b) or Eq. (10)
- J_n = Bessel function
- k = reduced frequency, $\omega b/U$
- k^* = k/β^2
- L = lift
- M = Mach number
- $\mathcal{M}$ = moment about midchord
- P = pressure
- $S(k)$ = Sears function
- t = time
- U = freestream velocity
- w = upwash
- x = axial coordinate
- y = coordinate normal to freestream
- β = $(1-M^2)^{1/2}$
- γ = Euler's constant
- ε = Mk
- θ = $-\cos^{-1}x$
- ρ = freestream density
- φ = $-\cos^{-1}x_1$
- ϕ = velocity potential

Received August 23, 1973; revision received October 4, 1973. This work was supported by the Pratt and Whitney Aircraft Division, Acoustics Group, under Engineering Order Supplement 730050-526.

Index categories: Nonsteady Aerodynamics; Subsonic and Transonic Flow.

* Research Engineer.

Fig. 1 Coordinate system for airfoil problem.

ω = circular frequency

Subscript

os = Osborne

Superscript

* = transformed variable, defined by Eq. (6)

Introduction

In a recent article, Osborne[1] presented an analytical approximation for the lift of an airfoil in a two-dimensional compressible subsonic flow with an imposed upwash which is sinusoidal in time (see Fig. 1). This solution was an attempt to include compressibility effects and was derived on the assumption that $(\varepsilon/\beta^2)^2 \ll 1$, where $\varepsilon \equiv kM$ is the product of the reduced frequency k and the Mach number M. (This product kM is proportional to the ratio of the airfoil chord to the acoustic wavelength.) However, Miles[2] has raised questions concerning the validity of applying the solution procedure used by Osborne to two-dimensional airfoil problems which have shed vorticity downstream. For this problem Miles[3] presents a solution which is an expansion for small values of k, neglecting terms of order k^2 and higher. (By order k^n we also imply order $k^n \log k$ throughout this Note.) The present Note examines the relation between the solution of Miles and that of Osborne. Also, by manipulation and modification of the Miles solution, an expression for the lift of an airfoil entering a Sears-type (spatially as well as temporally sinusoidal) gust in compressible flow is obtained. This result appears to be a significant improvement over both Osborne's result for this particular case and the original result of Miles when comparison is made with exact numerical solutions.[4,5]

The Osborne Solution

The equation for the perturbation potential $\phi(x, y, t)$ is

$$\left[\nabla^2 - M^2\left(\frac{k}{\omega}\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)^2\right]\phi(x, y, t) = 0 \qquad (1)$$

and the boundary conditions in the $y = 0$ plane are[2]

$$\begin{aligned} &\phi = 0 && x \leqq -1 \\ &(\partial\phi/\partial y) = bw(x)e^{i\omega t} && -1 < x \leqq 1 \\ &[(\partial/\partial t) + (U/b)\partial/\partial x]\phi = 0 && x \geqq 1 \end{aligned} \qquad (2)$$

The coordinates x and y have everywhere been nondimensionalized with the semichord b. A sinusoidal time dependence $e^{i\omega t}$ has been assumed and $w(x)$ represents the upwash distribution induced by the airfoil.

By transforming the coordinates and time according to the relations

$$x \to X; \quad y \to Y/\beta; \quad t \to (T - kM^2 X/\beta^2)/\omega \qquad (3)$$

Eqs. (1) and (2) become

$$(\nabla_{X,Y}{}^2 - M^2 k^{*2}\partial^2/\partial T^2)\phi^* = 0 \qquad (4)$$

$$\begin{aligned} &\phi^* = 0 && X \leqq -1 \\ &\partial\phi^*/\partial Y = (w^*/k^*)e^{iT} && -1 < X \leqq 1 \\ &(k^*\partial/\partial T + \partial/\partial X)\phi^* = 0 && X \geqq 1 \end{aligned} \qquad (5)$$

where

$$\begin{aligned} &k^* = k/\beta^2 \\ &w^*(X) = [w(x)/U\beta]\,e^{-ik^*M^2x} \\ &\phi^*(X, Y, T) = (\beta^2/\omega b^2)\phi(x, y, t) \end{aligned} \qquad (6)$$

When cast in this form it will be noted that except for the factor $M^2k^{*2}\partial^2\phi/\partial T^2$ appearing in Eq. (4), the formulation is exactly that of an incompressible problem. The argument then is that since $\nabla_{X,Y}{}^2\phi^*$ and $\partial^2\phi^*/\partial T^2$ have been nondimensionalized to be of the same order in the vicinity of the airfoil, the term $M^2k^{*2}\partial^2\phi^*/\partial T^2$ can be neglected compared to $\nabla_{X,Y}{}^2\phi^*$ and the solution will still be accurate including terms of order Mk^*.

The solution for the pressure distribution on the airfoil surface obtained through the similarity to the incompressible problem is

$$P_{os}(x, t) = \pm\frac{\rho U}{\pi\beta}e^{i(\omega t + k^*M^2x)}\int_0^\pi w(-\cos\varphi)\times e^{iM^2k^*\cos\varphi}g(\theta, \varphi, k^*)\,d\varphi \qquad (7a)$$

where

$$g(\theta, \varphi, k^*) = [\cos\varphi + C(k^*)(1-\cos\varphi)]\cot\frac{\theta}{2} + \frac{\sin\theta}{\cos\theta - \cos\varphi} + \tfrac{1}{2}k^*\sin\varphi\ln\left[\frac{1-\cos(\theta+\varphi)}{1-\cos(\theta-\varphi)}\right] \qquad (7b)$$

The $\pm$ in Eq. (7a) refers to the upper and lower airfoil surfaces, respectively. This result reduces to the proper incompressible result [quoted in Ref. 6, Eq. (5–19) for example] when $M \to 0$.

The Improved Solution

For three-dimensional (finite span) problems, Miles[2] gives an approximation procedure which is essentially the same as that described in the preceding section. In attempting to apply this procedure to two-dimensional problems, however, he states that "the solution for ϕ would involve integration over the infinite span, and this integration generally would not be commutative with the assumed expansion in powers of the frequency."

Because of this difficulty, Miles derived a separate solution for the two-dimensional case. By expansion of Possio's integral equation for the two-dimensional compressible airfoil problem in terms of reduced frequency and neglecting terms of order k^2 and higher, Miles derived the following relation for airfoil surface pressure for the two-dimensional problem:

$$P(x, t) = \pm\frac{\rho U}{\pi\beta}e^{i\omega t}\int_0^\pi w(-\cos\varphi)h(\theta, \varphi; k^*, M)\,d\varphi \qquad (8a)$$

where

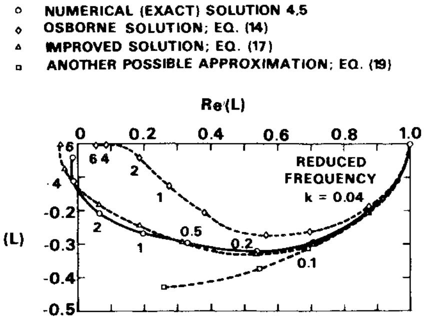

Fig. 2 Real and imaginary parts of the lift for an airfoil passing through a spatially sinusoidal gust, $M = 0.6$. Results normalized to 1 for $k = 0$, and gust referenced to airfoil leading edge.

$$h_1(\theta, \varphi; k^*, M) = \cot\frac{\theta}{2} + \frac{\sin\theta}{\cos\theta - \cos\phi} + \frac{i}{2}k^* \times \left\{\sin\varphi \ln\left[\frac{1-\cos(\theta+\varphi)}{1-\cos(\theta-\varphi)}\right] + 2\left[\gamma + i\frac{\pi}{2} + \ln\frac{k^*}{2} + f(M) - M^2\right] \times (1-\cos\varphi)\cot\frac{\theta}{2}\right\} + 0(k^{*2}) \quad (8b)$$

$$f(M) = (1-\beta)\ln M + \beta\ln(1+\beta) - \ln 2 \approx (M^2/2)\ln(M/2) - (M^2/4) \qquad M \ll 1$$

$$\gamma = \text{Euler's constant} = 0.5772\ldots$$

This result is the starting point for the result to be derived here.

One drawback of Eq. (8) is that it is accurate only for very low reduced frequency. One can get an idea of its range of accuracy by considering the small k expansion of the Theodorsen function $C(k)$ which is used in incompressible airfoil theory[6] and thus is related to the $M = 0$ limit of Eq. (8b).

$$C(k) = 1 - (\pi k/2) + ik[\ln(k/2) + \gamma] + 0(k^2) \quad (9)$$

Comparison with the exact values of $C(k)$ shows that the expansion given by Eq. (9) is accurate only for $k < 0.1$, and a similar range is expected for Eq. (8b). However, in his derivation Miles noted a similarity with the incompressible solution even when $M \neq 0$, a similarity which becomes apparent on comparison of Eqs. (8b) and (9). Neglecting no term of order lower than k^2, Eq. (8b) can be rewritten

$$h_2(\theta, \varphi; k^*, M) = g(\theta, \varphi, k^*) \times e^{ik^*[M^2 - f(M)](\cos\varphi - \cos\theta)} + 0(M^2k^{*2}) \quad (10)$$

which constitutes the improved solution. This form for $h(\theta, \varphi; k^*, M)$ becomes exact for all k when $M \to 0$. This statement would also hold true if the exponential in Eq. (10) were written as

$$e^{ik^*[M^2 - f(M)](\cos\varphi - \cos\theta)} \approx 1 + ik^*[M^2 - f(M)](\cos\varphi - \cos\theta) \quad (11)$$

which may be more convenient for computational purposes. When written in the form of Eq. (10), however, one notes a close similarity to the solution obtained by the Osborne analysis, i.e., setting $f(M) = 0$ reduces Eq. (10) [taken along with Eq. (8a)] to Eqs. (7).

By manipulation of the Miles solution into this form, the order of error has been reduced from k^{*2} in Eq. (8b) to M^2k^{*2} in Eq. (10). This can be seen to be true from the following argument. Based on the order of error given for Eq. (8b), the order of error for Eq. (10) can be written as $k^{*2}F(M)$ where $F(M)$ is some as yet undetermined function of the Mach number. The fact that Eq. (10) becomes exact for $M = 0$ implies that $F(M) \to 0$ as $M \to 0$. It will be noted, however, that only even powers of M appear in the problem formulation given by Eqs. (3–6), and an expansion in M of the exact solution would give only even powers such as M^{2n} and $M^{2n}\ln M$. Thus, $F(M)$ is of order M^2 and the order of error of Eq. (10) is M^2k^{*2}.

Sears Type Sinusoidal Gust

One case of particular interest is that of a sinusoidal gust drifting with the freestream. For this case

$$w(x) = w_0 e^{-ikx} \quad (12)$$

If this is substituted into Eqs. (7) obtained by the Osborne analysis, the result is

$$P_{os}(x,t) = \pm\frac{\rho}{\beta}Uw_0\left(\frac{1-x}{1+x}\right)^{1/2} S(k^*)\, e^{i(\omega t + k^*M^2x)} \quad (13)$$

The lift and moment are then

$$L_{os}(t) = -(2/\beta)\pi\rho U w_0 b S(k^*)[J_0(M^2k^*) - iJ_1(M^2k^*)]\, e^{i\omega t} \quad (14)$$

$$\mathscr{M}_{os}(t) = (2/\beta)\pi\rho U w_0 b^2 S(k^*)[J_0(M^2k^*) - iJ_1(M^2k^*) - J_1(M^2k^*)/M^2k^*]\, e^{i\omega t} \quad (15)$$

On the other hand, if the improved solution, Eq. (10), is used with Eqs. (8a) and (12), the following results accurate to order M^2k^{*2} can be derived:

$$P(x,t) = P_{os}(x,t)\, e^{ik^*f(M)} \quad (16)$$

$$L(t) = L_{os}(t)\, e^{ik^*f(M)} \quad (17)$$

$$\mathscr{M}(t) = \mathscr{M}_{os}(t)\, e^{ik^*f(M)} \quad (18)$$

These last three results were calculated by manipulating Eq. (10) along the lines suggested by Eq. (11) to arrive at a more easily integrable form while neglecting no terms of order lower than M^2k^{*2}. Equations (16–18) point out the close similarity between the Osborne solution and the improved solution.

It should be emphasized that these results are for two-dimensional flow. Miles[2,3] gives some discussion of the three-dimensional effects and concludes that the anomalous two-dimensional effects are important only for aspect ratios greater than 10.

Equation (17) for the lift on an airfoil passing through a sinusoidal gust is easily checked against available numerical solutions.[4,5] Figure 2 shows the real and imaginary parts of the lift for a Mach number of 0.6. The solution has been normalized to 1 for $k = 0$. It has also been multiplied by e^{-ik} so as to reference the gust to the airfoil leading edge rather than the midchord, thereby removing the spiral nature of the curve and making comparison between the various theories easier. It is evident from this figure that Eq. (17) agrees quite well with the numerical solutions for reduced frequencies smaller than 2, but this solution deteriorates rapidly for reduced frequencies larger than 2. The Osborne solution, however, is inaccurate above a reduced frequency of 0.2. Since the Osborne solution differs from Eq. (17) (the improved solution) only in phase, the Osborne solution is relatively accurate in amplitude but inaccurate in phase.

The accuracy of Eq. (17) should not be attributed solely to the fact that the derivation was accurate to (but not including) terms of order M^2k^{*2}. For example, another solution can be constructed which is identical to Eq. (17) to this order, i.e.,

$$L(t) = L_{os}(t) + ik^*f(M)L_{os}(t, k=0) \quad (19)$$

This expression is also plotted in Fig. 2 and it is obvious that Eq. (19) becomes inaccurate at a much lower reduced frequency than Eq. (17). One important difference between Eqs. (17) and (19) is that Eq. (17) behaves as k^{-1} for large k while Eq. (19) goes as k^{+1}. It can be shown[8,9] that the proper behavior for the lift at large k is k^{-1}. Of course, since Eqs. (17) and (19) are meant to be accurate only for $M^2k^{*2} \ll 1$, they would not be expected to be accurate for $k \gg 1$; but even though the constant multiplying k^{-1} in the large k expansion of Eq. (17) is incorrect, the fact that the lift given by Eq. (17) goes to zero rather than infinity as $k \to \infty$ may partially explain why Eq. (17) remains accurate to higher k values than Eq. (19).

Based on the results of Fig. 2, it appears that a reasonable criterion for the applicability of Eq. (17) is

$$Mk^* < 1 \quad (20)$$

Thus, as might be expected, the present solution becomes inapplicable when $M \to 1$. A solution for the transonic case has been derived by Landahl.[8] For high reduced frequency a solution such as that given by Adamczyk[9] for a semi-infinite flat plate can be used.

Summary and Conclusions

Based on a previous solution of Miles an analytical expression has been derived for the pressure, lift, and moment on a two-dimensional airfoil encountering a sinusoidal gust in compressible subsonic flow. The solution is closely related to the solution of Osborne and appears to give accurate results in the range $Mk^* < 1$. The close agreement in this range with the numerical solutions of Refs. 4 and 5 gives added confidence in the accuracy of both the numerical solutions and the analytic solution presented here.

References

[1] Osborne, C., "Unsteady Thin-Airfoil Theory for Subsonic Flow," *AIAA Journal*, Vol. 11, No. 2, Feb. 1973, pp. 205–209.

[2] Miles, J. W., "On the Compressibility Correction for Subsonic Unsteady Flow," *Journal of the Aeronautical Sciences*, Vol. 17, 1950, p. 181.

[3] Miles, J. W., "Quasi-Stationary Airfoil Theory in Subsonic Compressible Flow," *Quarterly of Applied Mathematics*, Vol. 8, 1950, pp. 350–358.

[4] Graham, J. M. R., "Similarity Rules for Thin Aerofoils in Non-Stationary Flows," *Journal of Fluid Mechanics*, Vol. 43, 1970, pp. 753–756.

[5] Adamczyk, J. J., "Passage of an Isolated Airfoil Through a Three-Dimensional Disturbance," Ph.D. thesis, 1971, Univ. of Connecticut, Storrs, Conn.; also Rept. UAR-K97, 1971, United Aircraft Corp., East Hartford, Conn.

[6] Garrick, I. E., "Non-Steady Wing Characteristics," *Aerodynamic Components of Aircraft at High Speeds*, edited by A. F. Donovan and H. R. Laurence, Princeton Univ. Press, Princeton, N.J., 1957.

[7] Miles, J. W., "A Note on Possio's Integral Equation for an Oscillating Airfoil in Subsonic Flow," *Quarterly of Applied Mathematics*, Vol. 7, 1949, pp. 213–217.

[8] Landahl, M. J., "Unsteady Transonic Flow," Pergamon Press, New York, 1961.

[9] Adamczyk, J. J., "Analytical Investigation of Compressibility and Three-Dimensionality on the Unsteady Response of an Airfoil in a Fluctuating Flow Field," AIAA Paper 73-683, Palm Springs, Calif., 1973.